變分原理與古典力學導論

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第一部分：引言

現代的物理學發展的框架下，喜歡從作用量 Action S出發，當物理學家寫下 Action 後（根據實驗、物理現象和限制等等猜出），針對 Action 做變分δS（Variation）後，在最小作用量原理（Principle of least action）$\delta S=0$的要求下，就可以得到物理遵守的運動方程，後續就根據不同的物理系統求滿足運動方程的物理量變化。

古典力學的發展上，由 d' Alembert 利用虛功原理可以得到

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}-\frac{\partial L}{\partial x_i}=0$$

後來 Hamilton 進一步闡明上述方程是滿足

$$S=\int_a^b L(x_1 \dots x_i,\dot{x}_1 \dots \dot{x}_i,t)dt$$

利用變分法取極值$\delta S=0$的必然結果。

在物理學上，變分法通常用於新理論在初期發展時，由於物理學家還不知道正確的運動方程，故會根據實驗成果、物理經驗等去猜 Action 可能的形式，然後利用變分法得到描述 Lagrangian 滿足的運動方程，此後變分法便功成身退，後續的解題過程只要處理運動方程即可。例如

$$S=\int_a^b L(x,\dot{x},t)dt$$

用變分後可以得到 Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-\frac{\partial L}{\partial x}=0$$

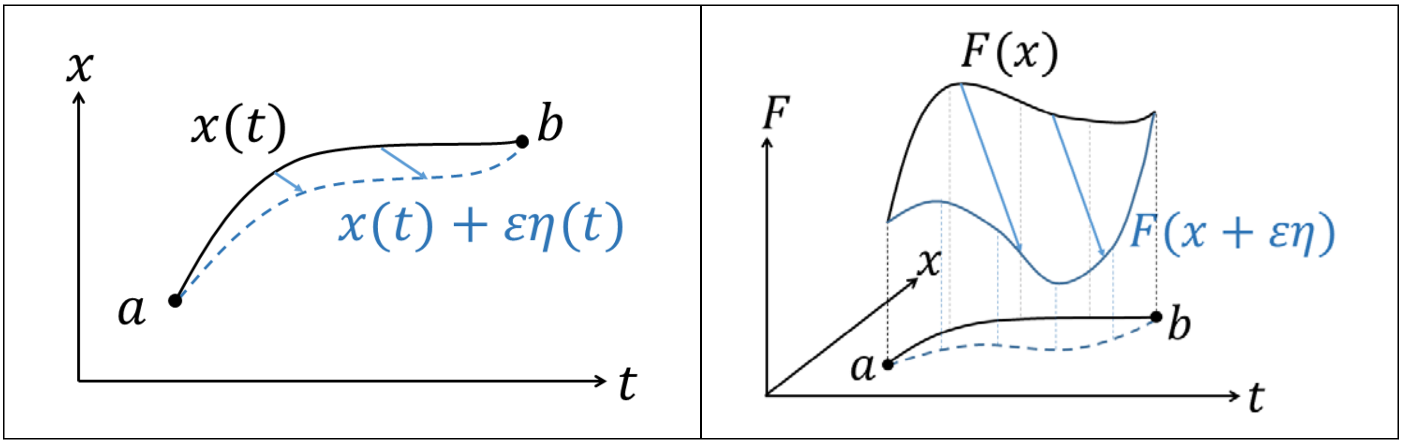
後續理論力學課程只要會解 Euler-Lagrange equation 就好，所以整學年的課程變分法不常出現，於是應數相關的書籍對於變分法提及就不多。故在此簡單用不是數學上嚴謹的方式稍為介紹一下變分法。

第二部分：初階變分法

如果今天，一個泛函（Functional）的積分問題

$$S[F(x,\dot{x},t)] = \int_a^b F(x,\dot{x},t)dt$$

其中，$x = x(t)$、$\dot{x} = \frac{dx}{dt}$，固定 a、b 下改變軌跡$x(t)$的形式會對積分S造成影響，我們想找到S的極值（Extreme value，不論極大或極小）下，軌跡$x(t)$會是什麼形式？或是$F(x,\dot{x},t)$該滿足什麼條件？根據微積分的概念，在$f(x)$極值x_0附近做微小的變化$x_0+\varepsilon$時，$f(x)$是不會有變化的，即$df=0$。類似的想法，S在極值附近時，$x(t)$稍微改變形式，$\delta S=0$。我們可以將積分問題簡單的用圖像表達，不同$x(t)$的函數形式表示不同的路徑連結 a 到 b 點。



如果$x(t)$是滿足S的極值，那麼如果加入微小的任意函數$\varepsilon\eta(t)$，其中$\eta(t)$滿足$\eta(a)=\eta(b)=0$，使 a、b 兩點$F(x,\dot{x},t)$不變。在加入微小的任意函數$\varepsilon\eta(t)$後

$$x(t) \to x(t) + \varepsilon\eta(t)$$
$$\dot{x}(t) \to \dot{x}(t) + \varepsilon\dot{\eta}(t)$$
$$F(x,\dot{x},t) \to F(x+\varepsilon\eta, \dot{x}+\varepsilon\dot{\eta}, t) = F(x,\dot{x},t) + \delta F$$

我們知道S在極值附近，微小變化ε時，S不變：

$$\delta S = S[F(x+\varepsilon\eta, \dot{x}+\varepsilon\dot{\eta}, t)] - S[F(x,\dot{x},t)] = 0 \leftrightarrow \frac{\delta S}{\delta\varepsilon} = 0$$

δS的變化完全是由δF造成的，所以可以寫為

$$\frac{\delta S}{\delta\varepsilon} = \frac{\delta}{\delta\varepsilon}\int_a^b F dt = \int_a^b \frac{\delta F}{\delta\varepsilon} dt = 0$$

利用微積分的手法，將$\frac{\delta F}{\delta\varepsilon}$展開

$$\frac{\delta S}{\delta\varepsilon} = \int_a^b \frac{\delta F}{\delta\varepsilon} dt = \int_a^b \frac{\partial F}{\partial x}\frac{\delta x}{\delta\varepsilon} + \frac{\partial F}{\partial \dot{x}}\frac{\delta \dot{x}}{\delta\varepsilon} + \frac{\partial F}{\partial t}\frac{\delta t}{\delta\varepsilon} dt$$

但我們只針對做軌跡x變分，沒有對t變分，所以

$$\frac{\delta t}{\delta \varepsilon}=0$$

$$\frac{\delta S}{\delta \varepsilon}=\int_a^b \frac{\partial F}{\partial x}\frac{\delta x}{\delta \varepsilon}+\frac{\partial F}{\partial \dot{x}}\frac{\delta \dot{x}}{\delta \varepsilon}dt$$

觀察$\frac{\delta x}{\delta \varepsilon}$、$\frac{\delta \dot{x}}{\delta \varepsilon}$：

$$\begin{cases}\dfrac{\delta x}{\delta \varepsilon}=\dfrac{\delta}{\delta \varepsilon}(x+\varepsilon\eta)=\eta \\ \dfrac{\delta \dot{x}}{\delta \varepsilon}=\dfrac{\delta}{\delta \varepsilon}(\dot{x}+\varepsilon\dot{\eta})=\dot{\eta}\end{cases}$$

可以得到

$$\frac{\delta S}{\delta \varepsilon}=\int_a^b \frac{\partial F}{\partial x}\eta+\frac{\partial F}{\partial \dot{x}}\dot{\eta}dt=\int_a^b \frac{\partial F}{\partial x}\eta+\frac{\partial F}{\partial \dot{x}}\frac{d\eta}{dt}dt=\int_a^b \frac{\partial F}{\partial x}\eta dt+\int_a^b \frac{\partial F}{\partial \dot{x}}\frac{d\eta}{dt}dt$$

我們針對最後一項做分部積分

$$\int_a^b \frac{\partial F}{\partial \dot{x}}\frac{d\eta}{dt}dt=\frac{\partial F}{\partial \dot{x}}\eta\Big|_a^b-\int_a^b \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\eta dt$$

注意紅色這一項，因為我們要求$\eta(t)$滿足$\eta(a)=\eta(b)=0$，所以$\frac{\partial F}{\partial \dot{x}}\eta\Big|_a^b=0$

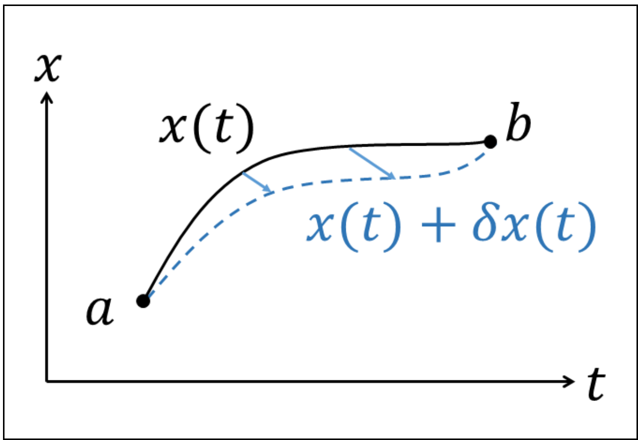
$$\therefore \frac{\delta S}{\delta \varepsilon}=\int_a^b \frac{\partial F}{\partial x}\eta dt-\int_a^b \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\eta dt=\int_a^b \left(\frac{\partial F}{\partial x}-\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\eta dt=0$$

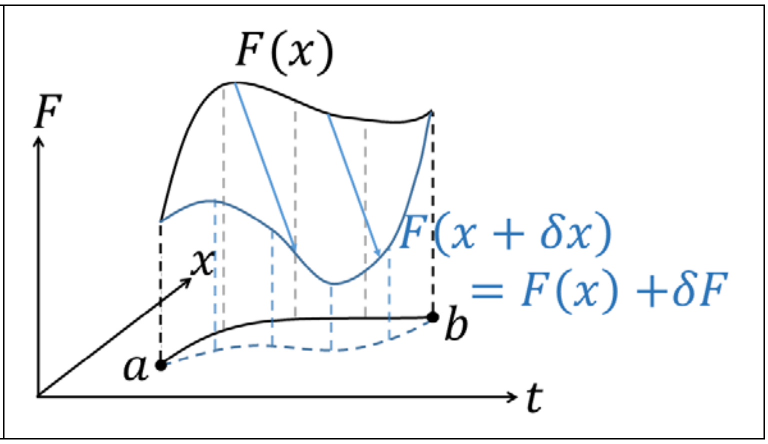
因為$\eta(t)$是任意的，所以

$$\frac{\partial F}{\partial x}-\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}=0\ \ Euler-Lagrange\ \ equation$$

第三部分：簡化變分表示法

第一部分介紹了簡單的變分法概念，但是需要引入任意的函數$\eta(t)$，手法上稍嫌煩瑣，不利於後續操作。第二部分以相同的概念，採用比較抽象的想法但相同的數學手法，演示一次變分法的操作。有點像將變分的操作類同於微分操作。





針對同一種泛函（Functional）的積分問題

$$S[F(x,\dot{x},t)] = \int_a^b F(x,\dot{x},t)dt$$

當我們針對x做變分，

$$x \to x + \delta x$$

變分δx滿足

$$\delta x(a) = \delta x(b) = 0$$

x的變分會導致$\dot{x}$、F發生變化

$$\begin{cases} \dot{x}(x) \to \dot{x}(x+\delta x) = \dot{x}(x) + \delta\dot{x} \\ F(x,\dot{x},t) \to F(x+\delta x, \dot{x}(x)+\delta\dot{x}, t) = F(x,\dot{x},t) + \delta F \end{cases}$$

我們要求$\delta S = 0$

$$\delta S = \delta\int_a^b F dt = \int_a^b \delta F dt = 0$$

概念上很好理解的是，δF的變化和δx、δx有關，所以將δF展開

$$\delta S = \int_a^b \delta F dt = \int_a^b \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial \dot{x}}\delta\dot{x} dt$$

進階：Thm.1 :微分與變分對調

如果今天微分與變分針對的對象不同，如對 t 微分$\frac{d}{dt}$、對x變分δx，則$\frac{d}{dt}$與δ可以對調。

$$\delta\dot{f} = \dot{f}(x+\delta x) - \dot{f}(x) = \frac{d}{dt}\big(f(x+\delta x) - f(x)\big) = \frac{d}{dt}\delta f$$

將$\delta\dot{x}$對調$\frac{d}{dt}\delta x$

$$\delta S = \int_a^b \frac{\partial F}{\partial x}\delta x dt + \int_a^b \frac{\partial F}{\partial \dot{x}}\left(\frac{d}{dt}\delta x\right)dt$$

同樣的手法對第二項做分部積分

$$\int_a^b \frac{\partial F}{\partial \dot{x}}\left(\frac{d}{dt}\delta x\right)dt = \frac{\partial F}{\partial \dot{x}}\delta x\Big|_a^b - \int_a^b \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\delta x dt$$

注意紅色這一項，因為我們要求變分δx滿足$\delta x(a)=\delta x(b)=0$，所以$\frac{\partial F}{\partial \dot{x}}\delta x\Big|_a^b=0$

$$\delta S=\int_a^b \frac{\partial F}{\partial x}\delta x dt-\int_a^b\left(\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\delta x dt=\int_a^b\left(\frac{\partial F}{\partial x}-\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\delta x dt=0$$

$$\delta x\ \ is\ \ arbitrary.$$

$$\frac{\partial F}{\partial x}-\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}=0\ \ Euler-Lagrange\ \ equation$$

進階：Thm.2: 變分的 Chain rule $$\delta(FG)=\frac{\partial(FG)}{\partial x}\delta x+\frac{\partial(FG)}{\partial \dot{x}}\delta \dot{x}=\left(\frac{\partial F}{\partial x}G+F\frac{\partial G}{\partial x}\right)\delta x+\left(\frac{\partial F}{\partial \dot{x}}G+F\frac{\partial G}{\partial \dot{x}}\right)\delta \dot{x}$$ $$=\left(\frac{\partial F}{\partial x}\delta x+\frac{\partial F}{\partial \dot{x}}\delta \dot{x}\right)G+F\left(\frac{\partial G}{\partial x}\delta x+\frac{\partial G}{\partial \dot{x}}\delta \dot{x}\right)=\ \ \delta F\cdot G+F\cdot\delta G$$
進階：Thm.3: 針對函數F同乘同除另一函數 G，不影響變分 $$\delta F=\delta\left(F\cdot\frac{G}{G}\right)=\delta(F\cdot G\cdot G^{-1})=\delta F\cdot G\cdot G^{-1}+F\cdot\delta G\cdot G^{-1}+F\cdot G\cdot\delta(G^{-1})$$ $$=\delta F+F\cdot\delta G\cdot G^{-1}+F\cdot G\cdot\left(-\frac{\delta G}{G^2}\right)=\delta F$$

第四部分：淺談 Particle 與 EM Field 交互作用下的 Action

以前我們學古典力學時，完整描述一個 Particle 只須寫下它的 Lagrangian

$$S=\int_a^b L dt=\int_a^b T-U\ \ dt=\int_a^b T\ \ dt+\int_a^b -U\ \ dt=S_P+S_{PF}$$

其中，動能項T可視為 Free particle 的 Action S_P，位能項U就是 Particle 和 Field 交互作用的 Action S_{PF}。在電磁學我們學到 Field 也有帶有動量、能量，所以完整描述電磁運動會包含 Field 的 Action S_F

$$S=S_P+S_{PF}+S_F$$

第五部分：Free Particle Action 的變分

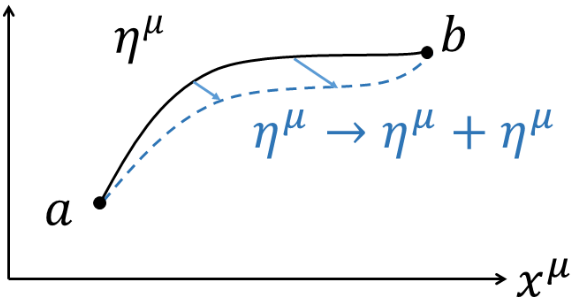
在相對論性下，描述 Free particle 我們會利用 4-displacement $\boldsymbol{\eta}=\eta^\mu\hat{e}_\mu=(\tau,\vec{0})_{porper}=(t,\vec{\eta})$來描述粒子的軌跡，其中$\tau$是 particle 的 proper time。這邊採用η^μ與x^μ區分軌跡與時空（因應後續諾特定理討論，需嚴謹區分軌跡和時空，軌跡是物理量，時空是座標，是不同的概念）；描述粒子速度利用 4-Velocity $\boldsymbol{U}=U^\mu\hat{e}_\mu=(\gamma c,\gamma\vec{v})=\frac{d\eta^\mu}{d\tau}\hat{e}_\mu$，其中$\tau$是 particle 的 proper time。更進一步說$U^\mu=U^\mu(x^\nu)$，速度U^μ會隨在時空的不同座標x^ν發生改變。在這邊的想法是，一個 Free particle 從時空中 a 跑到 b，我們針對不同路徑η^μ下的 Action S_P去算極值，即對x^μ作變分

$$\eta^\mu \to \eta^\mu + \delta\eta^\mu$$

$$\delta\eta^\mu(a) = \delta\eta^\mu(b) = 0$$

Free particle 的 Action S_P 為

$$S_P = \int_a^b -mc^2 d\tau$$



經過變分

$$\delta S_P = \delta\int_a^b -mc^2 d\tau = -mc^2\int_a^b \delta d\tau$$

看起來$d\tau$好像與$\eta^\mu \to \eta^\mu + \delta\eta^\mu$變分無關，不過回憶一件事情

$$\because c^2 d\tau^2 = d\eta^\mu d\eta_\mu$$

$$\therefore c d\tau = \sqrt{d\eta^\mu d\eta_\mu}$$

所以

$$\delta S_P = -mc\int_a^b \delta\sqrt{d\eta^\mu d\eta_\mu} = -mc\int_a^b \frac{1}{2}\frac{\delta d\eta^\mu \cdot d\eta_\mu + d\eta^\mu \cdot \delta d\eta_\mu}{\sqrt{d\eta^\mu d\eta_\mu}}$$

進階：Thm.4: 對 Scalar 變分與上下標無關

回憶度規張量 Metric Tensor $g_{\mu\nu}$：

$$g_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu$$

$$g_{\mu\nu} = g_{\nu\mu}$$

$$g^{\mu\nu} \equiv \left(g_{\mu\nu}\right)^{-1}$$

$$g^{\mu\nu}g_{\nu\omega} = \delta^\mu{}_\omega \ (Delta\ \ funcion\text{，暫時不要跟變分}\delta\text{搞混）。}$$

度規張量$g_{\mu\nu}$是時空的內稟性質(Intrinsic Property)，與x^μ無關，意思是對x^μ變分與度規$g_{\mu\nu}$無關。度規張量可以用作上下標轉換（Index lowering or raising）

$$x^\mu = g^{\mu\nu}x_\nu \text{、} x_\mu = g_{\mu\nu}x^\nu$$

因為對對x^μ變分與度規$g_{\mu\nu}$無關，所以

$$\delta x^\mu = g^{\mu\nu}\delta x_\nu \text{、} \delta x_\mu = g_{\mu\nu}\delta x^\nu$$

對一個 Scalar 作變分，例如$x^\mu y_\mu$是一個 Scalar（$x^\mu y_\mu = x_\mu y^\mu$）

$$\begin{aligned}\delta\left(x^\mu y_\mu\right) &= \delta x^\mu \cdot y_\mu + x^\mu \cdot \delta y_\mu = g^{\mu\nu}\delta x_\nu \cdot g_{\mu\omega}y^\omega + g^{\mu\nu}x_\nu \cdot g_{\mu\omega}\delta y^\omega \\ &= g^{\mu\nu}g_{\mu\omega}(\delta x_\nu \cdot y^\omega + x_\nu \cdot \delta y^\omega) = g^{\nu\mu}g_{\mu\omega}\delta(x_\nu y^\omega) \\ &= \delta^\nu{}_\omega \delta(x_\nu y^\omega) = \delta(x_\omega y^\omega) = \delta\left(x_\mu y^\mu\right)\end{aligned}$$

同理

$$x^\mu \delta y_\mu = x_\mu \delta y^\mu$$

所以

$$\delta dx^\mu \cdot dx_\mu + dx^\mu \cdot \delta dx_\mu = \delta dx^\mu \cdot dx_\mu + dx_\mu \cdot \delta dx^\mu = 2\delta dx^\mu \cdot dx_\mu$$

$$\delta S_P = -mc\int_a^b \frac{1}{2}\frac{2\delta d\eta^\mu \cdot d\eta_\mu}{\sqrt{dx^\mu dx_\mu}} = -mc\int_a^b \frac{\delta d\eta^\mu \cdot d\eta_\mu}{c d\tau} = -m\int_a^b \delta d\eta^\mu \cdot U_\mu$$

利用 *Thm.1* 的方法，我們將δdx^μ對調成$d\delta x^\mu$，並作分部積分

$$\delta S_P = -m\int_a^b U_\mu d\delta\eta^\mu = -U_\mu \delta\eta^\mu\Big|_a^b + m\int_a^b dU_\mu \delta\eta^\mu$$

邊界項因為變分邊界$\delta\eta^\mu(a) = \delta\eta^\mu(b) = 0$，後面那一項利用 *Thm.3* 同乘同除$d\tau$不影響變分

$$\delta S_P = m\int_a^b dU_\mu \delta\eta^\mu = \int_a^b m\frac{dU_\mu}{d\tau}\delta\eta^\mu d\tau$$

所以對 Free particle 而言，$\delta S_P = 0$使得

$$m\frac{dU_\mu}{d\tau} = 0$$

觀察$\mu = 1\sim3$

$$m\frac{d\vec{v}}{d\tau} = 0$$

Free particle 沒有加速度，保持等速運動。

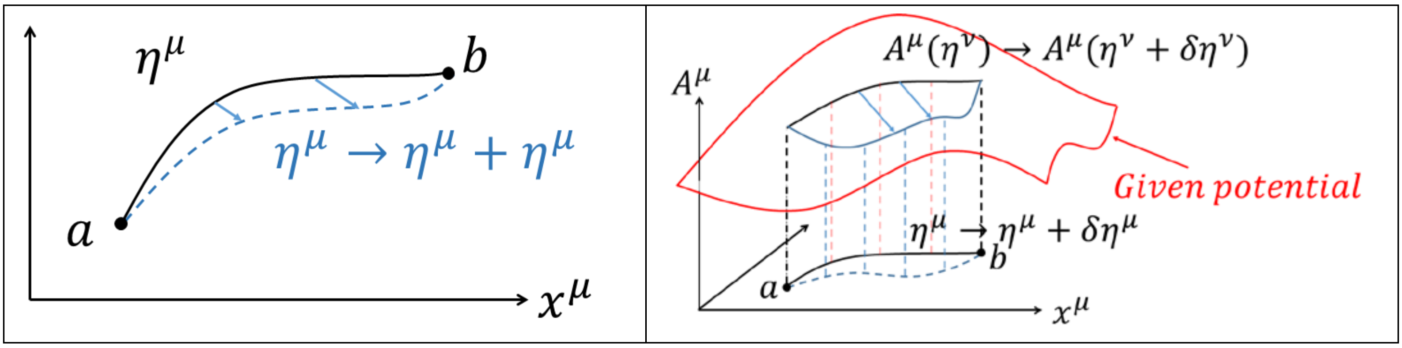
第六部分：Charge Particle 與 EM Field 作用下的變分與運動方程

在這一部分中，我們想探討一個 Charge Particle 在給定的 EM Field 下如何運動？（注意喔，給定的 EM Field 表示我們不對 EM Field 作變分）。相對論性電磁學下我們會寫下 4-Potential $\boldsymbol{A} = A^\mu \hat{e}_\mu = (\phi, \vec{A})$，在這邊採用高斯制（Gaussian unit），而 Charge Particle 交互作用的 Action S_{PF}會寫成

$$S_{PF} = \int_a^b -\frac{e}{c}A_\mu d\eta^\mu$$

完整的描述 Charge Particle 運動即為

$$S = S_P + S_{PF} = \int_a^b -mc^2 d\tau + \int_a^b -\frac{e}{c}A_\mu d\eta^\mu$$



在這邊，我們考慮 Charge Particle 在時空中的路徑作η^μ變分

$$\eta^\mu \to \eta^\mu + \delta\eta^\mu$$

雖然我們沒有對 EM Field A^μ作變分，但是走不同路徑感受到的位能是不一樣的，所以 Action 走不同的路徑會有不同的A^μ（意思是A^μ的變化來自於路徑x^μ不同，而不是對A^μ作變分）

$$A^\mu(\eta^\mu) \to A^\mu(\eta^\mu + \delta\eta^\mu) = A^\mu(\eta^\mu) + \delta A^\mu$$

我們計算δS_{PF}如何變分：

$$\delta S_{PF} = \delta\int_a^b -\frac{e}{c}A_\mu d\eta^\mu = -\frac{e}{c}\int_a^b \delta A_\mu \cdot d\eta^\mu - \frac{e}{c}\int_a^b A_\mu \cdot \delta d\eta^\mu$$

利用 *Thm.1* 的方法，我們將$\delta d\eta^\mu$對調$d\delta\eta^\mu$

$$\delta d\eta^\mu = d\delta\eta^\mu$$

並作分部積分，邊界項會消失

$$\delta S_{PF} = -\frac{e}{c}\int_a^b \delta A_\mu \cdot d\eta^\mu - \frac{e}{c}\int_a^b A_\mu \cdot d\delta\eta^\mu = -\frac{e}{c}\int_a^b \delta A_\mu \cdot d\eta^\mu - \frac{e}{c}A_\mu \cdot \delta\eta^\mu\Big|_a^b + \frac{e}{c}\int_a^b dA_\mu \cdot \delta\eta^\mu$$

利用 *Thm.3* 同乘同除$d\tau$不影響變分

$$\delta S_{PF}=-\frac{e}{c}\int_a^b \delta A_\mu\cdot d\eta^\mu+\frac{e}{c}\int_a^b dA_\mu\cdot\delta\eta^\mu=-\frac{e}{c}\int_a^b \delta A_\mu\cdot\frac{d\eta^\mu}{d\tau}d\tau+\frac{e}{c}\int_a^b\frac{dA_\mu}{d\tau}d\tau\cdot\delta\eta^\mu$$

因為

$$\begin{cases}\delta A_\mu=\dfrac{\partial A_\mu}{\partial\eta^\nu}\delta\eta^\nu\\ \dfrac{dA_\mu}{d\tau}=\dfrac{\partial A_\mu}{\partial x^\nu}\dfrac{d\eta^\nu}{d\tau}=\dfrac{\partial A_\mu}{\partial x^\nu}u^\nu\end{cases}$$

代入得到

$$\begin{aligned}\delta S_{PF}&=-\frac{e}{c}\int_a^b\frac{\partial A_\mu}{\partial x^\nu}\delta\eta^\nu\cdot u^\mu d\tau+\frac{e}{c}\int_a^b\frac{\partial A_\mu}{\partial x^\nu}u^\nu d\tau\cdot\delta\eta^\mu\\&=-\frac{e}{c}\int_a^b(\partial_\nu A_\mu)u^\mu\delta\eta^\nu d\tau+\frac{e}{c}\int_a^b(\partial_\nu A_\mu)u^\nu\delta\eta^\mu d\tau\end{aligned}$$

我們想要把變分$\delta\eta^\nu$和$\delta\eta^\mu$一起提出來，但是上標不一樣。但因為每一項μ、ν都是 Dummy index，可以互換$\mu\leftrightarrow\nu$，我們把第一項的μ、ν互換，就可以把兩項合併

$$\begin{aligned}\delta S_{PF}&=-\frac{e}{c}\int_a^b(\partial_\mu A_\nu)u^\nu\delta\eta^\mu d\tau+\frac{e}{c}\int_a^b(\partial_\nu A_\mu)u^\nu\delta\eta^\mu d\tau\\&=-\frac{e}{c}\int_a^b(\partial_\mu A_\nu-\partial_\nu A_\mu)u^\nu\delta\eta^\mu d\tau\end{aligned}$$

完整考慮 Charge Particle 在 EM Field 中的運動

$$\delta S=\delta S_P+\delta S_{PF}=0$$

所以

$$\begin{aligned}\delta S_P+\delta S_{PF}&=\int_a^b m\frac{dU_\mu}{d\tau}\delta\eta^\mu d\tau-\frac{e}{c}\int_a^b(\partial_\mu A_\nu-\partial_\nu A_\mu)u^\nu\delta\eta^\mu d\tau\\&=\int_a^b\left[m\frac{dU_\mu}{d\tau}-\frac{e}{c}(\partial_\mu A_\nu-\partial_\nu A_\mu)U^\nu\right]\delta\eta^\mu d\tau=0\end{aligned}$$

會得到

$$m\frac{dU_\mu}{d\tau}-\frac{e}{c}(\partial_\mu A_\nu-\partial_\nu A_\mu)U^\nu=0$$

$$m\frac{dU_\mu}{d\tau}=\frac{e}{c}(\partial_\mu A_\nu-\partial_\nu A_\mu)U^\nu\equiv\frac{e}{c}F_{\mu\nu}U^\nu$$

我們定義電磁張量 Electromagnetic Tensor

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

電場$\vec{E}$、磁場$\vec{B}$與ϕ、$\vec{A}$的關係（高斯制）

$$\begin{cases}\vec{E}=-\nabla\phi-\dfrac{1}{c}\dfrac{\partial\vec{A}}{\partial t}\\ \vec{B}=\nabla\times\vec{A}\end{cases}$$

其中

$$A_\nu\to(\phi,-A_x,-A_y,-A_z)$$

$$\partial_\mu\to\left(\frac{\partial}{\partial x^0},\frac{\partial}{\partial x^1},\frac{\partial}{\partial x^2},\frac{\partial}{\partial x^3}\right)=\left(\frac{\partial}{\partial ct},\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

可以計算

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

第七部分：4-Volume d^4x與 Lagrangian Density $\mathcal{L}$

因為時空在相對論下是等價的，利用相對性原理（Principle of Relativity）和最小作用量原理（Principle of least action）的要求，物理學家會將 Action S寫成 Scalar 的形式，從而保證在任何作標系下$\delta S = 0$。剛剛我們所列下來的 Action：

$$S_P = -mc^2 \int_a^b d\tau$$

$$S_{PF} = -\frac{e}{c}\int_a^b A_\mu dx^\mu$$

$$S = S_P + S_{PF} = -mc^2\int_a^b d\tau - \frac{e}{c}\int_a^b A_\mu dx^\mu = \int_a^b -\gamma mc^2 - \gamma e\phi + \frac{e}{c}\vec{A}\cdot\gamma\vec{v}\ \ dt = \int_a^b L dt$$

雖然 Action 都滿足 Scalar 的要求，但是 Lagrangian L本身並不是 Scalar，因為換到不同座標系下會不一樣，物理學家於是想要進一步將 Lagrangian L改寫成 Scalar 的形式。我們定義 4-Volume $d^4x = dc\tau dV = dc\tau dxdydz$，並將原本的 Action 改寫

$$S_P = -mc^2\int_a^b d\tau = -\int \rho_m dV\, c^2 \int_a^b d\tau = -\iint \rho_m c dc\tau dV = \int -\rho_m c d^4x$$

$$S_{PF} = -\frac{e}{c}\int_a^b A_\mu dx^\mu = -\frac{\int \rho dV}{c}\int_a^b A_\mu \frac{dx^\mu}{d\tau} d\tau = -\frac{1}{c}\iint \rho A_\mu u^\mu d\tau dV = -\frac{1}{c^2}\iint A_\mu J^\mu dc\tau d^4x$$

$$= -\frac{1}{c^2}\int A_\mu J^\mu d^4x$$

其中，4-current density $\boldsymbol{J} = J^\mu \hat{e}_\mu = \rho u^\mu \hat{e}_\mu$。特別的是，$d^4x$是一個不變量 Invariant，所以是一個 Scalar。另外在加上 EM Field 的 Action S_F

$$S_F = -\frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu} d^4x$$

$$S = S_P + S_{PF} + S_F = \int -\rho_m c - \frac{1}{c^2}A_\mu J^\mu - \frac{1}{16\pi c}F_{\mu\nu}F^{\mu\nu}\ \ d^4x \equiv \int \mathcal{L} d^4x$$

因為 4-Volume d^4x是一個 Scalar，Action 也是一個 Scalar，所以$\mathcal{L}$也是一個 Scalar。$\mathcal{L}$我們稱為 Lagrangian density，Lagrangian density　$\mathcal{L}$在任何座標系下都是 Scalar，形式保持不變：

$$\mathcal{L} = -\rho_m c - \frac{1}{c^2}A_\mu J^\mu - \frac{1}{16\pi c}F_{\mu\nu}F^{\mu\nu}$$

進階：d^4x是一個不變量 Invariant

物理 proof

因為 Time dilation 和 Length contraction 相反。如果τ、$\bar{x}$是 proper time 和 proper length

$$t = \gamma\tau$$

$$x = \frac{\bar{x}}{\gamma}$$

所以

$$dctdx = dc(\gamma\tau)d\left(\frac{\bar{x}}{\gamma}\right) = dc\tau d\bar{x}$$

數學 proof

回憶 Jacobian J

$$dxdy = rdrd\theta = J(r,\theta)drd\theta$$

其中

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial rcos\theta}{\partial r} & \frac{\partial rcos\theta}{\partial \theta} \\ \frac{\partial rsin\theta}{\partial r} & \frac{\partial rsin\theta}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r$$

同理

$$dc\bar{t}d\bar{x} = J(ct,x)dctdx = \begin{vmatrix} \frac{\partial c\bar{t}}{\partial ct} & \frac{\partial c\bar{t}}{\partial x} \\ \frac{\partial \bar{x}}{\partial ct} & \frac{\partial \bar{x}}{\partial x} \end{vmatrix} dctdx$$

回憶勞倫茲轉換

$$\begin{cases} c\bar{t} = \gamma(ct - \beta x) \\ \bar{x} = \gamma(x - \beta ct) \end{cases}$$

代回去計算

$$dc\bar{t}d\bar{x} = \begin{vmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{vmatrix} dctdx = \gamma^2(1-\beta^2)dctdx = dctdx$$

所以d^4x在勞倫茲轉換下是一個不變量。

第八部分：給定 Source 下對 EM Field 變分與 Maxwell equation

在這一部分，我們在給定 Source 下，討論 EM Field 的分佈（給定 Source 下表示我們不對 Source 下的分佈η^μ作變分）。

$$S = S_{PF} + S_F = \int -\frac{1}{c^2} A_\mu J^\mu d^4x + \int -\frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu} d^4x$$

我們想知道 EM Field 的分佈，所以我們針對A^μ作變分

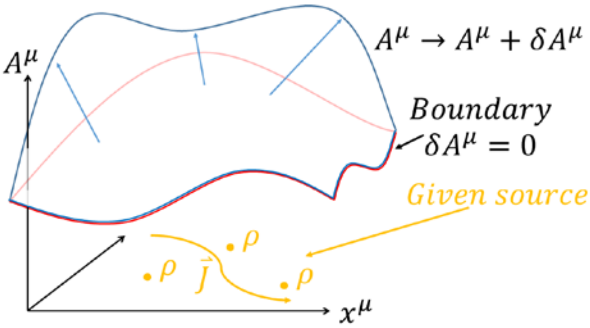
$$A^\mu \to A^\mu + \delta A^\mu$$

我們來觀察J^μ、d^4x、$F^{\mu\nu}$會不會受到影響？

$$J^\mu = \rho \frac{dx^\mu}{d\tau} = J^\mu(x^\nu)$$

$$d^4x = d^4x(x^\nu)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}(A^\omega)$$



可見只有電磁張量$F^{\mu\nu}$會受到A^μ的變分$A^\mu \to A^\mu + \delta A^\mu$有關

$$F^{\mu\nu}(A^\omega) \to F^{\mu\nu}(A^\omega + \delta A^\omega) = F^{\mu\nu}(A^\omega) + \delta F^{\mu\nu}$$

所以S_{PF}的變分很簡單

$$\delta S_{PF} = \delta \int -\frac{1}{c^2} A_\mu J^\mu d^4x = -\frac{1}{c^2} \int \delta A_\mu \cdot J^\mu d^4x$$

至於S_F的變分就稍嫌複雜

$$S_F = -\frac{1}{16\pi c} \int F_{\mu\nu} F^{\mu\nu} d^4x$$

$$\delta S_F = -\frac{1}{16\pi c} \delta \int F_{\mu\nu} F^{\mu\nu} d^4x = -\frac{1}{16\pi c} \int \delta F_{\mu\nu} \cdot F^{\mu\nu} + F_{\mu\nu} \cdot \delta F^{\mu\nu} d^4x$$

利用 *Thm.4:* 對 Scalar 變分與上下標無關，所以$\delta F_{\mu\nu} \cdot F^{\mu\nu} = F_{\mu\nu} \cdot \delta F^{\mu\nu}$，會有兩倍

$$\delta S_F = -\frac{1}{16\pi c} \int 2\delta F_{\mu\nu} \cdot F^{\mu\nu} d^4x = -\frac{1}{8\pi c} \int \delta F_{\mu\nu} \cdot F^{\mu\nu} d^4x$$

$$= -\frac{1}{8\pi c} \int \delta\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \cdot F^{\mu\nu} d^4x$$

$$= -\frac{1}{8\pi c} \int \delta\left(\partial_\mu A_\nu\right) \cdot F^{\mu\nu} d^4x + \frac{1}{8\pi c} \int \delta\left(\partial_\nu A_\mu\right) \cdot F^{\mu\nu} d^4x$$

因為每一項μ、ν都是 Dummy index，可以互換$\mu \leftrightarrow \nu$，我們把第一項的μ、ν互換，就可以把兩項合併

$$\delta S_F = -\frac{1}{8\pi c} \int \delta\left(\partial_\nu A_\mu\right) \cdot F^{\nu\mu} d^4x + \frac{1}{8\pi c} \int \delta\left(\partial_\nu A_\mu\right) \cdot F^{\mu\nu} d^4x$$

$$= \frac{1}{8\pi c} \int \delta\left(\partial_\nu A_\mu\right) \cdot \left(-F^{\nu\mu} + F^{\mu\nu}\right) d^4x$$

回憶電磁張量

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$F^{\mu\nu}$是一個反對稱張量，所以$F^{\nu\mu} = -F^{\mu\nu}$

代回去會多兩倍

$$\delta S_F = \frac{1}{8\pi c}\int \delta\left(\partial_\nu A_\mu\right)\cdot\left(F^{\mu\nu} + F^{\mu\nu}\right)d^4x = \frac{1}{4\pi c}\int \delta\left(\partial_\nu A_\mu\right)\cdot F^{\mu\nu}d^4x$$

利用 *Thm.1* 的方法，我們將δ和∂_ν對調

$$\delta\left(\partial_\nu A_\mu\right) = \partial_\nu\left(\delta A_\mu\right)$$

$$\delta S_F = \frac{1}{4\pi c}\int \partial_\nu\left(\delta A_\mu\right)\cdot F^{\mu\nu}d^4x$$

利用微分的 Chain rule，

$$\partial_\nu\left(\delta A_\mu\right)\cdot F^{\mu\nu} = \partial_\nu\left(\delta A_\mu\cdot F^{\mu\nu}\right) - \delta A_\mu\cdot\partial_\nu\left(F^{\mu\nu}\right)$$

將積分拆成兩項

$$\delta S_F = \frac{1}{4\pi c}\int \partial_\nu\left(\delta A_\mu\cdot F^{\mu\nu}\right)d^4x - \frac{1}{4\pi c}\int \delta A_\mu\cdot\partial_\nu\left(F^{\mu\nu}\right)d^4x$$

回憶 Divergence theorem

$$\int \nabla\cdot\vec{F}dV = \oint \vec{F}\cdot d\vec{S}$$

一個體積分，可以改寫成對體表面的面積分

寫成 Levi-Civita symbol

$$\int \partial_\nu F^\nu dV = \oint F^\nu dS_\nu$$

所以第一項利用 Divergence theorem，但是 Boundary 上的$\delta A_\mu = 0$，所以

$$\frac{1}{4\pi c}\int \partial_\nu\left(\delta A_\mu\cdot F^{\mu\nu}\right)d^4x = \frac{1}{4\pi c}\oint \delta A_\mu\cdot F^{\mu\nu}dS_\nu = 0$$

所以

$$\delta S_F = -\frac{1}{4\pi c}\int \partial_\nu\left(F^{\mu\nu}\right)\cdot\delta A_\mu d^4x$$

合併$\delta S_{PF} + \delta S_F$

$$\delta S_{PF} + \delta S_F = -\frac{1}{c^2}\int J^\mu\cdot\delta A_\mu d^4x - \frac{1}{4\pi c}\int \partial_\nu\left(F^{\mu\nu}\right)\cdot\delta A_\mu d^4x$$

$$= \int\left[-\frac{1}{c^2}J^\mu - \frac{1}{4\pi c}\partial_\nu F^{\mu\nu}\right]\delta A_\mu d^4x = 0$$

會得到

$$-\frac{1}{c^2}J^\mu - \frac{1}{4\pi c}\partial_\nu F^{\mu\nu} = 0$$

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c}J^\mu$$

$$Maxwell\ \ eq\ \ (Gaussian\ \ Unit)\begin{cases} \nabla\cdot\vec{E} = 4\pi\rho \\ \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \\ \nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \end{cases}$$

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} J^\mu \rightarrow \begin{cases} \nabla \cdot \vec{E} = 4\pi\rho \\ \nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \end{cases}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

電磁張量有特別的關係式

$$\partial_\omega F_{\mu\nu} + \partial_\mu F_{\nu\omega} + \partial_\nu F_{\omega\mu} = 0$$

展開

$$\partial_\omega(\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\mu(\partial_\nu A_\omega - \partial_\omega A_\nu) + \partial_\nu(\partial_\omega A_\mu - \partial_\mu A_\omega) = 0$$

$$\partial_\omega \partial_\mu A_\nu - \partial_\omega \partial_\nu A_\mu + \partial_\mu \partial_\nu A_\omega - \partial_\mu \partial_\omega A_\nu + \partial_\nu \partial_\omega A_\mu - \partial_\nu \partial_\mu A_\omega = 0$$

這條關係式會得到

$$\partial_\omega F_{\mu\nu} + \partial_\mu F_{\nu\omega} + \partial_\nu F_{\omega\mu} = 0 \rightarrow \begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \end{cases}$$

第九部分：電磁學中的規範不變性 Gauge Invariance

在電磁學中，我們定義 Potential ϕ、$\vec{A}$和電場$\vec{E}$、磁場$\vec{B}$的關係：

$$\begin{cases} \vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\ \vec{B} = \nabla \times \vec{A} \end{cases}$$

但 Potential ϕ、$\vec{A}$不唯一，可以引入一個 Gauge function $G(ct, \vec{x})$

$$\begin{cases} \phi' = \phi + \frac{1}{c}\frac{\partial G}{\partial t} \\ \vec{A}' = \vec{A} - \nabla G \end{cases}$$

一樣保持電場$\vec{E}$、磁場$\vec{B}$不變

$$\begin{cases} \vec{E}' = -\nabla\phi' - \frac{1}{c}\frac{\partial \vec{A}'}{\partial t} - \nabla\phi - \nabla\left(\frac{1}{c}\frac{\partial G}{\partial t}\right) - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \frac{1}{c}\frac{\partial(\nabla G)}{\partial t} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = \vec{E} \\ \vec{B}' = \nabla \times \vec{A}' = \nabla \times \vec{A} - \nabla \times \nabla G = \nabla \times \vec{A} = \vec{B} \end{cases}$$

Note $\nabla \times \nabla G = 0$

$$(\nabla \times \nabla G)_i = \varepsilon_{ijk}\partial_j\partial_k G = \frac{1}{2}\varepsilon_{ijk}\partial_j\partial_k G + \frac{1}{2}\varepsilon_{ijk}\partial_j\partial_k G = \frac{1}{2}\varepsilon_{ijk}\partial_j\partial_k G + \frac{1}{2}\varepsilon_{ikj}\partial_j\partial_k G$$

$$= \frac{1}{2}\varepsilon_{ijk}\partial_j\partial_k G - \frac{1}{2}\varepsilon_{ijk}\partial_j\partial_k G = 0$$

在相對論中，上述的操作可以寫為 4-Potential 的形式：

$$(\phi', \vec{A}') = (\phi, \vec{A}) + (\partial_{ct} G, -\nabla G) = (\phi, \vec{A}) + (\partial_{ct}, -\nabla) G$$

回憶

$$\partial^\mu = (\partial_{ct}, -\nabla)$$

故寫成

$$A'^\mu = A^\mu + \partial^\mu G$$

Coulomb gauge condition：非相對論性 Non-relativistic $\nabla\cdot\vec{A}=0$
Lorenz gauge condition：相對論性，滿足 Lorentz transformation $\nabla\cdot\vec{A}+\frac{1}{c}\frac{\partial\phi}{\partial t}=0$ 寫成 Scalar form $\partial_\mu A^\mu=0$ 所以滿足相對論轉換。

剛剛已經寫下電場$\vec{E}$、磁場$\vec{B}$在加入 Gauge 之下保持不變

$$\begin{cases}\vec{E}'=\vec{E}\\ \vec{B}'=\vec{B}\end{cases}$$

這其實說明，$F^{\mu\nu}$也在加入 Gauge 之下保持不變：

$$F'^{\mu\nu}=F^{\mu\nu}$$

當然我們也可以寫下證明

$$\begin{aligned}F'^{\mu\nu}&=\partial^\mu A'^\nu-\partial^\nu A'^\mu=\partial^\mu(A^\nu+\partial^\nu G)-\partial^\nu(A^\mu+\partial^\mu G)\\&=\partial^\mu A^\nu+\partial^\mu\partial^\nu G-\partial^\nu A^\mu-\partial^\nu\partial^\mu G\\&=\partial^\mu A^\nu-\partial^\nu A^\mu=F^{\mu\nu}\end{aligned}$$

第十部分：規範不變與 Action 的變分

這部分要討論的是，加入 Gauge 不會改變 Action 的變分，意味著運動方程不會受到 Gauge 的改變，也代表 Gauge Invariance。當加入 Gauge 時。：

$$A'^\mu=A^\mu+\partial^\mu G$$

我們來觀察一下 Action $S=S_P+S_{PF}+S_F$有誰會受到影響：

$$S_P=-mc^2\int_a^b d\tau=\int-\rho_m c\,d^4x$$

$$S_{PF}=-\frac{e}{c}\int_a^b A_\mu dx^\mu=-\frac{1}{c^2}\int A_\mu J^\mu d^4x$$

$$S_F=\int-\frac{1}{16\pi c}F_{\mu\nu}F^{\mu\nu}d^4x$$

因為剛剛已經證明，$F^{\mu\nu}$不受到 Gauge 的影響，而S_P沒有A^μ相關，所以唯一受到影響的是S_{PF}

$$S'_{PF}=-\frac{e}{c}\int_a^b A'_\mu dx^\mu=-\frac{1}{c^2}\int A'_\mu J^\mu d^4x$$

針對S'_{PF}變分

$$\delta S'_{PF}=-\frac{1}{c^2}\delta\int A'_\mu J^\mu d^4x=-\frac{1}{c^2}\delta\int A_\mu J^\mu d^4x-\frac{1}{c^2}\delta\int\partial_\mu G\cdot J^\mu d^4x$$

將後面那一項利用微分的 Chain rule

$$\partial_\mu G\cdot J^\mu=\partial_\mu(GJ^\mu)-G\partial_\mu J^\mu$$

$$\delta S'_{PF}=-\frac{1}{c^2}\delta\int A_\mu J^\mu d^4x-\frac{1}{c^2}\delta\int\partial_\mu(GJ^\mu)d^4x+\frac{1}{c^2}\delta\int G\partial_\mu J^\mu d^4x$$

中間項利用 Divergence theorem 改寫成

$$\delta S'_{PF}=-\frac{1}{c^2}\delta\int A_\mu J^\mu d^4x-\frac{1}{c^2}\delta\left[\oint GJ^\mu dS_\mu\right]+\frac{1}{c^2}\delta\int G\left[\partial_\mu J^\mu\right]d^4x$$

中間項因為 Boundary 上的變分為 0，所以中間項為 0：

$$\delta\left[\oint GJ^\mu dS_\mu\right]=0$$

最後一項回憶（由實驗觀察到的）連續性方程 Continuity equation

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\vec{J}=0$$

寫成相對論性

$$\partial_\mu J^\mu=0$$

導致最後一項也為 0

$$\delta\int G\left[\partial_\mu J^\mu\right]d^4x=0$$

所以

$$\delta S'_{PF}=-\frac{1}{c^2}\delta\int A_\mu J^\mu d^4x=\delta S_{PF}$$

這表示加入 Gauge 不會改變 Action 的變分，整套物理滿足 Gauge Invariance。

第十一部分：淺談 Gauge Invariance 和 Continuity equation

剛剛我們的推導中，利用連續性方程 Continuity equation 得到 Action 的變分滿足規範不變性。而 Continuity equation 對應到的是電荷守恆 Charge conservation。在物理的發展後期，物理學開始反其道而行，當我們要求 Action 滿足規範不變性，便可以得到 Continuity equation 和 Charge conservation。變分的過程完全一樣，唯一的差別在於我們強迫

$$\frac{1}{c^2}\delta\int G\left[\partial_\mu J^\mu\right]d^4x=0$$

以保證變分結果不受到 Gauge 影響，這就可以得到

$$\partial_\mu J^\mu=0$$

學到這部分，大概可以了解到，物理學家常常說 Action 強大的地方，我們可以寫下 Action 就可以得到實驗上所有得到的實驗方程。但真正 Action 的發展上，都必須要有實驗先得到部分或全部的運動方程，物理學家經由一些物理論證，猜出 Action 的形式，再藉由 Action 去得到其餘更深刻的物理。由於古典電磁學的發展很全面，電磁學裡面的所有方程都已經經由實驗得到，所以在推導 Action 的變分得到$F^{\mu\nu}$或 Maxwell eq 就覺得很像看著答案寫問題。不過物理的發展上我們實驗上得到電荷守恆，物理學家改寫成 Action 的描述，可以諾特定理 Noether's theorem 很嚴謹的連結電荷守恆和規範不變性的關係。這就像是過去從牛頓力學經由虛功原理可以得到 Action 的描述（Hamilton principle），物理學家便可以反過來從 Action 得到牛頓力學，但是 Action 整套東西還可以來描述電磁學，甚至是後來的量子力學（Feynman 的 Path integral），Action 對於推廣物理有至關重要的地位。

第十二部分：Impossibility of $A_\mu A^\mu$ if keeping gauge invariance

如果保持規範不變的話，我們可以論證 Action 不會有$A_\mu A^\mu$項，因為A^μ的變分$A^\mu\rightarrow A^\mu+\delta A^\mu$會導致運動方程變化。

$$S=-\frac{1}{c^2}\int A_\mu A^\mu d^4x$$

加入 Gauge $\partial^\mu G$

$$A'^\mu = A^\mu + \partial^\mu G$$

展開 Action

$$S' = -\frac{1}{c^2}\int A'_\mu A'^\mu d^4x = -\frac{1}{c^2}\int (A_\mu + \partial_\mu G)(A^\mu + \partial^\mu G) d^4x$$

$$= -\frac{1}{c^2}\int A_\mu A^\mu d^4x - \frac{1}{c^2}\int 2A^\mu \cdot \partial_\mu G d^4x - \frac{1}{c^2}\int \partial_\mu G \cdot \partial^\mu G d^4x$$

因為 Gauge $\partial^\mu G$獨立於 4-Potential A^μ，A^μ的變分$A^\mu \to A^\mu + \delta A^\mu$不影響$\partial^\mu G$。我們比較對$S$變分和對$S'$變分的差異：

對S變分

$$\delta S = -\frac{1}{c^2}\delta\int A_\mu A^\mu d^4x = -\frac{2}{c^2}\int A_\mu \delta A^\mu d^4x$$

對S'變分

$$\delta S' = -\frac{1}{c^2}\delta\int A_\mu A^\mu d^4x - \frac{1}{c^2}\delta\int 2A_\mu \partial^\mu G d^4x - 0\ (\partial^\mu G\text{沒有變分})$$

第一項就是原本的變分

$$\delta S' = -\frac{2}{c^2}\int A_\mu \delta A^\mu d^4x - \frac{2}{c^2}\delta\int A^\mu \cdot \partial_\mu G d^4x$$

$$= -\frac{2}{c^2}\int (A_\mu + \partial_\mu G)\delta A^\mu d^4x$$

比對後會發現兩者不一樣，會導致規範不變被破壞。如果我們希望保持規範不變性，那就不會出現$A_\mu A^\mu$。當然，也許某一天實驗發現規範不變性是錯誤的，那有可能可以引入$A_\mu A^\mu$項。

第十三部分：淺談量子場論中的$A_\mu A^\mu$-光子質量

本篇內容主要在古典範疇討論，在此簡單討論一下$A_\mu A^\mu$在量子場論中對應到的是光子質量。$Klein–Gordon\ equation$描述自旋整數的粒子，由來簡單的從相對論出發，根據$4 - momentum$：

$$P_\mu P^\mu = m^2c^2$$

量子場論與古典的關係其中一點就是所有物理量都改寫成算符(operator)作用到 wave function ϕ

$$\hat{P}_\mu \hat{P}^\mu \phi = m^2c^2\phi$$

量子場論中動量算符為

$$\hat{P}^\mu = i\hbar\partial^\mu$$

代入

$$(i\hbar\partial_\mu)(i\hbar\partial^\mu)\phi = m^2c^2\phi$$
$$-\hbar^2\partial_\mu\partial^\mu\phi = m^2c^2\phi$$

得到$Klein–Gordon\ equation$

$$(\hbar^2\partial_\mu\partial^\mu + m^2c^2)\phi = 0$$

如果設$\hbar = c = 1$（普朗克單位制或自然單位制）

$$(\Box + m^2)\phi = 0$$

就是常見的形式。在量子場論中描述 Spin 1 massive particle 的$Lagrangian\ dendity$:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}m^2 A_\nu A^\nu (\text{採用 } \hbar = c = 1)$$

對δA^{υ}變分滿足

$$\partial^{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial^{\mu}A^{\upsilon})}\right)-\frac{\partial\mathcal{L}}{\partial A^{\upsilon}}=0$$

得到

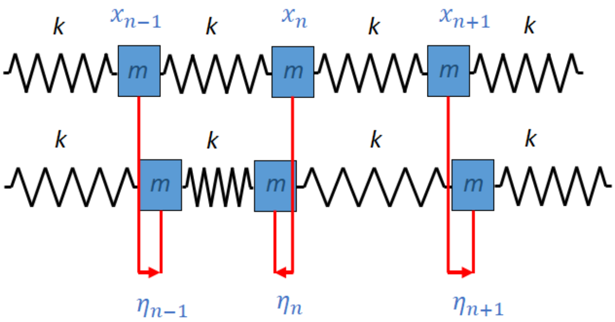
$$-\partial^{\mu}(\partial_{\mu}A_{\upsilon})-m^2A_{\upsilon}=0\rightarrow\partial^{\mu}\partial_{\mu}A_{\upsilon}+m^2A_{\upsilon}=0$$
$$(\Box+m^2)A_{\upsilon}=0$$

就得到$Klein\text{–}Gordon\ \ equation$描述 Spin 1 massive particle 的 EOM。可以看到$A_{\upsilon}A^{\upsilon}$對應到的是光子質量。不過，正確描述 Spin 1 massless particle 的$Lagrangian\ \ dendity$並不是$\mathcal{L}=-\frac{1}{2}\partial_{\mu}A_{\upsilon}\partial^{\mu}A^{\upsilon}$，而是

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

後續涉及到光子無質量造成的一些自由度問題，會需要一些技術細節將$4-potential$的四個自由度消除兩個，以對應到光子只有兩個偏振方向，就不再此討論。

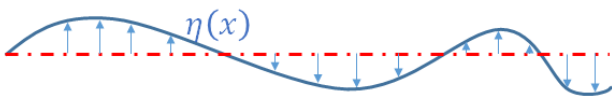
第十四部分：連續場與波動方程



從單質點進入到連續場之中，物理學家是利用串聯無窮多的簡諧振子去描述一個連續場。針對無窮多的簡諧振子，可以寫下$Lagrangian$

$$L=\sum_{n}^{\infty}\frac{1}{2}m\dot{\eta}_n-\frac{1}{2}k(\eta_n-\eta_{n-1})^2$$

其中η_n是位置 n 的振幅，$\dot{\eta}_n$是位置 n 的振盪的速度。



進入到連續場時，$Lagrangian$為

$$L=\int\frac{1}{2}\rho\left(\frac{\partial\eta}{\partial t}\right)^2-\frac{1}{2}Y\left(\frac{\partial\eta}{\partial x}\right)^2dx$$

其中ρ為質量密度、Y為楊氏係數。

在三維$Lagrangian$為

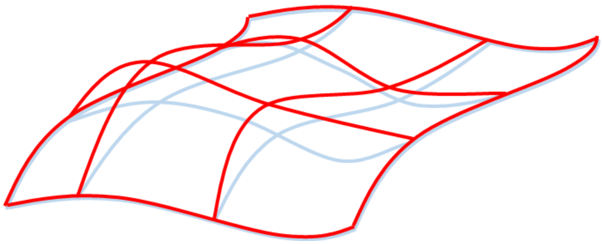
$$L=\int\frac{1}{2}\rho(\partial_t\eta)^2-\frac{1}{2}Y(\nabla\eta)^2dV$$

可以直接看出$Lagrangian\ dendity\ \mathcal{L}=\frac{1}{2}\rho(\partial_t\eta)^2-\frac{1}{2}Y(\nabla\eta)^2$。一樣的，我們可以利用 Variational principle 得到連續場的運動方程。作用量 Action 為

$$S=\int\mathcal{L}d^4x=\int\frac{1}{2}\rho(\partial_t\eta)^2-\frac{1}{2}Y(\nabla\eta)^2d^4x$$

將場振幅做變分 $\eta(x^\mu)\rightarrow\eta(x^\mu)+\delta\eta(x^\mu)$，Action 變化要為極值：

$$\eta(x^\mu)\rightarrow\eta(x^\mu)+\delta\eta(x^\mu)$$



$$\delta S=\delta\int\frac{1}{2}\rho(\partial_t\eta)^2-\frac{1}{2}Y(\nabla\eta)^2d^4x=\int\rho\partial_t\eta*\delta(\partial_t\eta)-Y\nabla\eta*\delta(\nabla\eta)d^4x$$

將變分和微分交換

$$=\int\rho\partial_t\eta\partial_t(\delta\eta)-Y\nabla\eta\nabla(\delta\eta)d^4x=\int\rho\partial_t\eta\partial_t(\delta\eta)d^4x-\int Y\nabla\eta\nabla(\delta\eta)d^4x$$

$$=\int\partial_t(\rho\partial_t\eta\delta\eta)d^4x-\int\partial_t(\rho\partial_t\eta)*\delta\eta d^4x-\int\nabla(Y\nabla\eta\delta\eta)d^4x+\int\nabla(Y\eta)*\delta\eta d^4x$$

$$=\left[-\int\partial_t(\rho\partial_t\eta)*\delta\eta d^4x+\int\nabla(Y\eta)*\delta\eta d^4x\right]+\left[\int\partial_t(\rho\partial_t\eta\delta\eta)d^4x-\int\nabla(Y\nabla\eta\delta\eta)d^4x\right]$$

後面為邊界項，利用 Integral by part

$$=-\int[\rho\partial_t^2\eta-Y\nabla^2\eta]*\delta\eta d^4x+\left[\int\rho\partial_t\eta\delta\eta d^3x\Big|_{t_1}^{t_2}-\int Y\nabla\eta\delta\eta d^3x\Big|_{\partial x,\partial y,\partial z}\right]$$

$$=-\int[\rho\partial_t^2\eta-Y\nabla^2\eta]*\delta\eta d^4x+0$$

因為$\delta\eta$為任意，得到

$$\rho\partial_t^2\eta-Y\nabla^2\eta=0$$

$$\nabla^2\eta=\frac{\rho}{Y}\partial_t^2\eta$$

令$v=\sqrt{\frac{Y}{\rho}}$

$$\nabla^2\eta=\frac{1}{v^2}\partial_t^2\eta$$

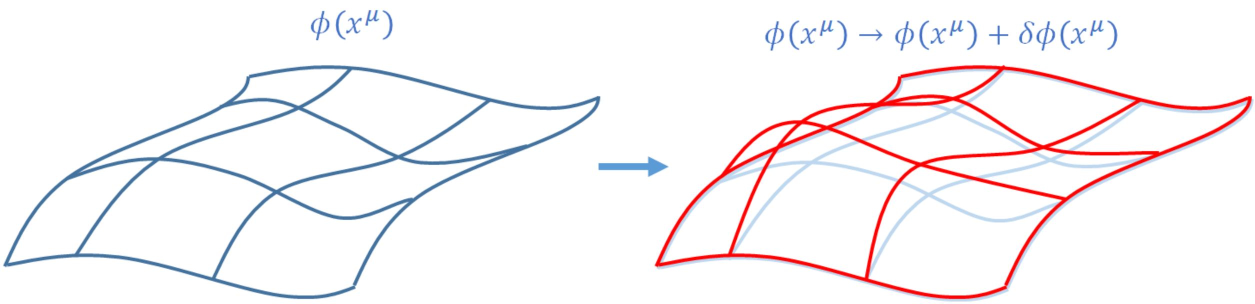
即波動方程。

第十五部分：古典場論

在古典場論中，物理學將連續場的振幅η推廣到古典場的物理量，可以是純量場ϕ、向量場V^{μ}、張量場$T^{\mu\upsilon}$等等。先可慮簡單的純量場$\phi=\phi(x^{\mu})$是時空函數，通過 Variational principle 找到ϕ的 EOM。先寫下 Action，假設$Lagrangian\ \ dendity\ \ \mathcal{L}$是$\phi,\partial_{\mu}\phi$的函數，並通過$\phi\to\phi+\delta\phi$的變分

$$S=\int\mathcal{L}[\phi,\partial_{\mu}\phi]d^4x$$

$$\phi\to\phi+\delta\phi\text{、}\partial_{\mu}\phi\to\partial_{\mu}\phi+\delta\partial_{\mu}\phi$$



$$\delta S=\delta\int\mathcal{L}[\phi,\partial_{\mu}\phi]d^4x=\int\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi+\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta\partial_{\mu}\phi\ \ d^4x$$

$$=\int\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi-\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\right)\delta\phi\ \ d^4x+\int\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta\phi\right)\ \ d^4x$$

$$=\int\left[\frac{\partial\mathcal{L}}{\partial\phi}-\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\right)\right]\delta\phi\ \ d^4x+\int\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta\phi\ \ d^3x\bigg|_{\partial V}$$

$$=\int\left[\frac{\partial\mathcal{L}}{\partial\phi}-\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\right)\right]\delta\phi\ \ d^4x+0$$

因為$\delta\phi$是任意，所以對任何古典物理量ϕ，必須滿足$Euler-Lagrange\ \ eq$

$$\frac{\partial\mathcal{L}}{\partial\phi}-\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\right)=0$$

同樣的，我們可以回到連續場的例子，$\mathcal{L}=\frac{1}{2}\rho(\partial_t\eta)^2-\frac{1}{2}Y(\nabla\eta)^2$，滿足

$$\frac{\partial\mathcal{L}}{\partial\phi}-\partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\phi)}\right)-\nabla\left(\frac{\partial\mathcal{L}}{\partial(\nabla\phi)}\right)=0$$

可以迅速得到

$$-\rho\partial_t^2\eta+Y\nabla^2\eta=0\to\nabla^2\eta=\frac{1}{v^2}\partial_t^2\eta$$

如果我們將純量場ϕ推廣成向量場，例如$4-displacement\ \ x^{\upsilon}$、$4-potential\ \ A^{\upsilon}$，剛剛的變分只需將$\phi$改寫成$A^{\upsilon}$，即可得

$$\frac{\partial\mathcal{L}}{\partial x^{\upsilon}}-\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial U^{\upsilon}}\right)=0\ \ Lorentz\ \ force$$

$$\frac{\partial\mathcal{L}}{\partial A^{\upsilon}}-\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A^{\upsilon})}\right)=0\ \ Field\ \ equation$$

回到電磁場的 Action

$$S=S_P+S_{PF}+S_F=-\int P_\mu U^\mu d\tau-\int\frac{e}{c}A_\mu U^\mu d\tau-\int\frac{1}{16\pi c}F_{\mu\nu}F^{\mu\nu}\ \ d^4x$$

$$=\int -\rho_m c-\frac{1}{c^2}A_\mu J^\mu-\frac{1}{16\pi c}F_{\mu\nu}F^{\mu\nu}\ \ d^4x$$

先計算$Lorentz\ \ force$

$$\frac{\partial\mathcal{L}}{\partial x^\upsilon}-\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial U^\upsilon}\right)=0$$

跟x^υ有關的只有S_P 、S_{PF}，所以處理$\mathcal{L}=-P_\mu U^\mu-\frac{e}{c}A_\mu U^\mu$

$$\frac{\partial\left(-\frac{e}{c}A_\mu U^\mu\right)}{\partial x^\upsilon}-\frac{d}{d\tau}\left(\frac{\partial}{\partial U^\upsilon}\left(-P_\mu U^\mu-\frac{e}{c}A_\mu U^\mu\right)\right)=0\rightarrow-\frac{e}{c}U^\mu\left(\partial_\upsilon A_\mu\right)-\frac{d}{d\tau}\left(-P_\upsilon-\frac{e}{c}A_\upsilon\right)=0$$

$$-\frac{e}{c}U^\mu\left(\partial_\upsilon A_\mu\right)+\frac{dP_\upsilon}{d\tau}+\frac{e}{c}\frac{dA_\upsilon}{d\tau}=0\rightarrow-\frac{e}{c}U^\mu\left(\partial_\upsilon A_\mu\right)+\frac{dP_\upsilon}{d\tau}+\frac{e}{c}\frac{\partial A_\upsilon}{\partial x^\mu}\frac{dx^\mu}{d\tau}=0$$

$$\rightarrow-\frac{e}{c}U^\mu\left(\partial_\upsilon A_\mu\right)+\frac{dP_\upsilon}{d\tau}+\frac{e}{c}U^\mu\left(\partial_\mu A_\upsilon\right)=0\rightarrow\frac{dP_\upsilon}{d\tau}+\frac{e}{c}U^\mu\left(\partial_\mu A_\upsilon-\partial_\upsilon A_\mu\right)=0$$

$$\rightarrow\frac{dP_\upsilon}{d\tau}+\frac{e}{c}U^\mu F_{\mu\upsilon}=0\rightarrow\frac{dP_\mu}{d\tau}=-\frac{e}{c}U^\upsilon F_{\upsilon\mu}\rightarrow\frac{dP_\mu}{d\tau}=\frac{e}{c}F_{\mu\upsilon}U^\upsilon$$

迅速得到$Lorentz\ \ force$。再來處理$Field\ \ equation$，

$$\partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu A^\upsilon)}\right)-\frac{\partial\mathcal{L}}{\partial A^\upsilon}=0$$

$$\partial^\mu\left[-\frac{1}{8\pi c}F_{\alpha\beta}\frac{\partial F^{\alpha\beta}}{\partial(\partial^\mu A^\upsilon)}\right]+\frac{1}{c^2}J_\upsilon=0$$

因為$\frac{\partial F^{\alpha\beta}}{\partial(\partial^\mu A^\upsilon)}=\frac{\partial}{\partial(\partial^\mu A^\upsilon)}\left(\partial^\alpha A^\beta-\partial^\beta A^\alpha\right)=\delta^\alpha{}_\mu\delta^\beta{}_\upsilon-\delta^\beta{}_\mu\delta^\alpha{}_\upsilon$

$$\partial^\mu\left[-\frac{1}{8\pi c}F_{\alpha\beta}\left(\delta^\alpha{}_\mu\delta^\beta{}_\upsilon-\delta^\beta{}_\mu\delta^\alpha{}_\upsilon\right)\right]+\frac{1}{c^2}J_\upsilon=0$$

$$\partial^\mu\left[-\frac{1}{8\pi c}F_{\mu\upsilon}+\frac{1}{8\pi c}F_{\upsilon\mu}\right]+\frac{1}{c^2}J_\upsilon=0$$

因為$-F_{\mu\upsilon}=F_{\upsilon\mu}$

$$\partial^\mu\left[\frac{1}{4\pi c}F_{\upsilon\mu}\right]+\frac{1}{c^2}J_\upsilon=0$$

$$\partial_\nu F^{\mu\nu}=-\frac{4\pi}{c}J^\mu$$

第十六部分：諾特定理 Noether theorem-General proof

提到變分法就一定要與諾特定理做搭配。諾特定理表明如果系統中$Lagrangian$的物理量ϕ存在連續變換$\phi\rightarrow\phi+\delta\alpha$，使得$\mathcal{L}\rightarrow\mathcal{L}+\frac{\delta\mathcal{L}}{\delta\alpha}$，卻保持對稱性（不變性）$\frac{\delta\mathcal{L}}{\delta\alpha}=0$，那麼就存在一個守恆量與之對應。這可以應用到：

時間平移不變 ↔ 能量守恆

空間平移不變 ↔ 動量守恆

廣義的證明如下，給定$Lagrangian\ \ dendity\ \ \mathcal{L}\left[\phi(x^\alpha),\partial_\mu\phi(x^\alpha),x^\mu\right]$，之前的變分法是針對物理量$\phi$做變分，現在我們同時針對物理量$\phi$和時空$x^\mu$變分

$$x^\mu \to x'^\mu = x^\mu + \delta x^\mu$$
$$\phi \to \phi' = \phi + \delta\phi$$

但是$\phi = \phi(x^\alpha)$是時空的函數，經過物理量ϕ和時空x^μ同時變分後，包含自身的變分和受到時空變分的影響$\phi(x^\alpha) \to \phi'\left(x'^\alpha\right)$，令$\phi'\left(x'^\alpha\right) \equiv \phi(x^\alpha) + \Delta\phi$，

$$\Delta\phi = \phi'\left(x'^\alpha\right) - \phi(x^\alpha) = \phi'\left(x'^\alpha\right) - \phi\left(x'^\alpha\right) + \phi\left(x'^\alpha\right) - \phi(x^\alpha) = \delta\phi + \frac{\partial\phi}{\partial x^\alpha}\delta x^\alpha$$

諾特定理要求通過變分後 Action 不變，$\delta S = 0$。

$$\delta S = \delta\int \mathcal{L}\left[\phi(x^\alpha),\partial_\mu\phi(x^\alpha),x^\mu\right]d^4x$$

$$= \int \mathcal{L}\left[\phi'\left(x'^\alpha\right),\partial_\mu\phi'\left(x'^\alpha\right),x'^\mu\right]d^4x' - \int \mathcal{L}\left[\phi(x^\alpha),\partial_\mu\phi(x^\alpha),x^\mu\right]d^4x$$

很明顯可以發現到連積分的$4-\text{volune}\ \ d^4x'$都發生變化。可以用乘法的微分來思考

$$\delta S = \delta\int \mathcal{L}d^4x = \int \delta\mathcal{L}\ \ d^4x + \int \mathcal{L}\ \ \delta d^4x$$

第一項$\delta\mathcal{L}$在處理時要非常小心

$$\delta\mathcal{L} = \mathcal{L}\left[\phi'\left(x'^\alpha\right),\partial_\mu\phi'\left(x'^\alpha\right),x'^\mu\right] - \mathcal{L}\left[\phi(x^\alpha),\partial_\mu\phi(x^\alpha),x^\mu\right]$$
$$= \mathcal{L}\left[\phi'\left(x'^\alpha\right),\partial_\mu\phi'\left(x'^\alpha\right),x'^\mu\right] - \mathcal{L}\left[\phi\left(x'^\alpha\right),\partial_\mu\phi\left(x'^\alpha\right),x'^\mu\right] + \mathcal{L}\left[\phi\left(x'^\alpha\right),\partial_\mu\phi\left(x'^\alpha\right),x'^\mu\right] - \mathcal{L}\left[\phi(x^\alpha),\partial_\mu\phi(x^\alpha),x^\mu\right]$$
$$= \left\{\mathcal{L}\left[\phi+\delta\phi,\partial_\mu\phi+\delta\left(\partial_\mu\phi\right),x'^\mu\right] - \mathcal{L}\left[\phi,\partial_\mu\phi,x'^\mu\right]\right\} + \left\{\mathcal{L}\left[\phi,\partial_\mu\phi,x'^\mu\right] - \mathcal{L}\left[\phi,\partial_\mu\phi,x^\mu\right]\right\}$$
$$= \left\{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\left(\partial_\mu\phi\right)\right\} + \left\{\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu\right\}$$

利用∂_μ拉到前面去

$$= \left\{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right)\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi\right)\right\} + \left\{\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu\right\}$$

可以注意到$\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right)\right]\delta\phi$是$Euler-Lagrange\ \ eq$等於零。所以

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\delta\phi\right) + \partial_\mu\mathcal{L} * \delta x^\mu$$

第二項δd^4x要用到$Jacobian$

$$\int \mathcal{L}\ \ \delta d^4x = \int \mathcal{L}\ \ d^4x' - \int \mathcal{L}\ \ d^4x$$

$$d^4x' = J\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)d^4x$$

$J\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)$為$Jacobian\ \ determinant$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \frac{\partial(x^\mu + \delta x^\mu)}{\partial x^\nu} = \delta^\mu{}_\nu + \partial_\nu\delta x^\mu$$

$Jacobian\ \ determinant$計算出來為[1]

$$J\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) = 1 + \partial_\mu\delta x^\mu$$

[1] "Quantum Field Theory", Ryder, P83

所以

$$\int \mathcal{L}\ \ \delta d^4x = \int \mathcal{L} * \partial_\mu \delta x^\mu \ \ d^4x$$

合併兩項

$$\delta S = \delta \int \mathcal{L} d^4x = \int \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi\right) + \partial_\mu \mathcal{L} * \delta x^\mu\right] d^4x + \int \left[\mathcal{L} * \partial_\mu \delta x^\mu\right]\ \ d^4x$$

$$= \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi\right) + \partial_\mu \mathcal{L} * \delta x^\mu + \mathcal{L} * \partial_\mu \delta x^\mu \ \ d^4x$$

$$= \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi\right) + \partial_\mu (\mathcal{L} \delta x^\mu)\ \ d^4x$$

$$= \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \mathcal{L} \delta x^\mu\right)\ \ d^4x = 0$$

其中$\delta\phi$是ϕ自身的變分，不包含受到δx^μ的影響，我們想要考慮廣義一點，改寫成$\Delta\phi$可以包含受到δx^μ的影響。因為$\Delta\phi = \delta\phi + \partial_\alpha \phi * \delta x^\alpha$，所以

$$\delta S = \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\Delta\phi - \partial_\alpha \phi * \delta x^\alpha) + \mathcal{L} \delta x^\mu\right)\ \ d^4x$$

$$= \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi - \delta^\mu{}_\alpha \mathcal{L}\right) \delta x^\alpha\right)\ \ d^4x$$

令

$$\begin{pmatrix} \delta x^\alpha \\ \Delta\phi \end{pmatrix} = \varepsilon \begin{pmatrix} \mathbf{X}^{\boldsymbol{\alpha}} \\ \boldsymbol{\Psi} \end{pmatrix}$$

$\mathbf{X}^{\boldsymbol{\alpha}}$、$\boldsymbol{\Psi}$為 Symmetry generators

$$\delta S = \varepsilon \int \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \boldsymbol{\Psi} - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi - \delta^\mu{}_\alpha \mathcal{L}\right) \mathbf{X}^{\boldsymbol{\alpha}}\right)\ \ d^4x = 0$$

因為ε是任意的，所以

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \boldsymbol{\Psi} - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi - \delta^\mu{}_\alpha \mathcal{L}\right) \mathbf{X}^{\boldsymbol{\alpha}}\right) \equiv \partial_\mu N^\mu = 0$$

定義諾特流($Noether\ \ current$)

$$N^\mu \equiv \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \boldsymbol{\Psi} - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi - \delta^\mu{}_\alpha \mathcal{L}\right) \mathbf{X}^{\boldsymbol{\alpha}}\right)$$

如果是向量場A^υ，

$$N^\mu \equiv \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathrm{A}^\upsilon)} \boldsymbol{\Psi}^{\boldsymbol{\upsilon}} - \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \mathrm{A}^\upsilon)} \partial_\alpha \mathrm{A}^\upsilon - \delta^\mu{}_\alpha \mathcal{L}\right) \mathbf{X}^{\boldsymbol{\alpha}}\right)$$

其中，$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$是對應到物理量$\phi$在對稱變換$\boldsymbol{\Psi}$的守恆流，$\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi - \delta^\mu{}_\alpha \mathcal{L}\right)$是對應時空$x^\mu$變化$\mathbf{X}^{\boldsymbol{\alpha}}$下的守恆流。

第十七部分：諾特定理與對稱性-時空與能量動量守恆

在討論時空對稱性時，我們固定物理量ϕ不受變分影響，$\mathbf{\Psi} = \mathbf{0}$時，只剩下

$$\partial_\mu\left(\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\alpha\phi - \delta^\mu{}_\alpha\mathcal{L}\right)\mathbf{X}^\alpha\right) = 0$$

我們定義能量動量張量$Energy\ \ momentum\ \ tensor\ \ T^{\mu\upsilon}$

$$T^{\mu\upsilon} \equiv \frac{\partial\mathcal{L}}{\partial(\partial_\upsilon\phi)}\partial^\mu\phi - \delta^{\mu\upsilon}\mathcal{L}$$

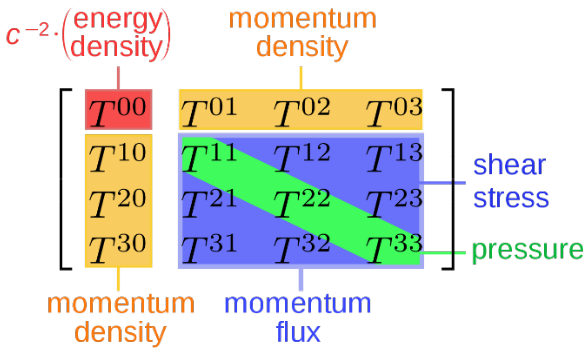
守恆流$\partial_\upsilon T^{\mu\upsilon} = 0$分別代表

$$\begin{cases}\partial_\upsilon T^{0\upsilon} = 0 \ \text{能量守恆} \\ \partial_\upsilon T^{i\upsilon} = 0 \ \text{動量守恆}\end{cases}$$

在相對論中，一個自由粒子的能量動量張量為

$$T^{\mu\upsilon} \equiv mU^\mu U^\upsilon$$

其中



對古典粒子來說，回到非相對論性的話，粒子的軌跡q只與t有關，

$$T^{\mu\upsilon} = T^{00} = \frac{\partial L}{\partial\dot{q}}\dot{q} - L = H$$

T^{00}就是$Hamiltonian$，能量守恆律

$$\partial_\upsilon T^{0\upsilon} = \partial_0 T^{00} = \frac{dH}{dt} = 0$$

能量不隨時間改變！古典的諾特定理表述為

古典非相對論性	相對論性
$S = \int Ldt$	$S = \int \mathcal{L}d^4x$
$N \equiv \left(\frac{\partial L}{\partial\dot{q}_i}\mathbf{Q}_i - \left(\frac{\partial L}{\partial\dot{q}_i}\dot{q}_i - L\right)\mathbf{T}\right)$	$N^\mu \equiv \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\upsilon)}\mathbf{\Psi}^\upsilon - \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\upsilon)}\partial_\alpha A^\upsilon - \delta^\mu{}_\alpha\mathcal{L}\right)\mathbf{X}^\alpha\right)$

古典中，時間t佔的角色對應於時空x^μ，所以時間的對稱性代表能量守恆，其餘得的對稱性$\mathbf{Q}_i$可是空間q_i、角度θ_i，分別是

$$\frac{\partial L}{\partial\dot{q}_i}\ \text{動量守恆}$$

$$\frac{\partial L}{\partial\dot{\theta}_i}\ \text{角動量守恆}$$